Remarks on the method of comparison equations (generalized WKB method) and the generalized Ermakov-Pinney equation

Alexander Kamenshchik [a,b], Mattia Luzzi [a] and Giovanni Venturi [a]

[a] *Dipartimento di Fisica and INFN, via Irnerio 46, 40126 Bologna, Italy*

[b] *L.D. Landau Institute for Theoretical Physics, Kosygin str. 2, 119334 Moscow, Russia*

Abstract

The connection between the method of comparison equations (generalized WKB method) and the Ermakov-Pinney equation is established. A perturbative scheme of solution of the generalized Ermakov-Pinney equation is developed and is applied to the construction of perturbative series for second-order differential equations with and without turning points.

Key words: WKB method, Ermakov-Pinney equation, perturbative scheme
PACS: 02.30.Hq, 03.65.Sq

1 Introduction: the general case

The WKB method is one of the main tools in quantum mechanics [1,2] and quantum field theory [3]. From the mathematical point of view this method coincides with the construction of the asymptotic expansions for the solution to the second-order differential equations with the small coefficient of the second derivative term [4,5]. In this note we consider an interesting modification of the WKB method - the so called method of comparison Eqs. [6,7] and analyse its relation with the Ermakov - Pinney equation [8,9]. Let us consider the second-order differential equation

Email addresses: kamenshchik@bo.infn.it (Alexander Kamenshchik), luzzi@bo.infn.it (Mattia Luzzi), armitage@bo.infn.it (Giovanni Venturi).

$$\frac{d^2u(x)}{dx^2} = \frac{1}{\varepsilon}\omega^2(x)u(x) , \tag{1}$$

where ε is a small parameter. Let us suppose that one knows the solution of another second-order differential equation

$$\frac{d^2U(\sigma)}{d\sigma^2} = \frac{1}{\varepsilon}\Omega^2(\sigma)U(\sigma) . \tag{2}$$

In this case one can represent an *exact* solution of the Eq. (1) in the form

$$u(x) = \left(\frac{d\sigma}{dx}\right)^{-1/2} U(\sigma) , \tag{3}$$

where the relation between the variables x and σ is given by the equation

$$\omega^2(x) = \left(\frac{d\sigma}{dx}\right)^2 \Omega^2(\sigma) + \varepsilon \left(\frac{d\sigma}{dx}\right)^{1/2} \frac{d^2}{dx^2} \left(\frac{d\sigma}{dx}\right)^{-1/2} . \tag{4}$$

On solving Eq. (4) for $\sigma(x)$ and substituting it into Eq. (3) one finds the solution to Eq. (1). Eq. (4) can be solved by using some iterative scheme. The application of such a scheme is equivalent to the application of the WKB method or, in other words, to the construction of the uniform asymptotic expansion for the solution of Eq. (1). The method of construction of the solution to Eq. (1) by means of the solutions of Eqs. (2) and (4) is called the method of comparison equations and the function $\Omega^2(\sigma)$ is called the comparison function [7].

The iterative scheme of solution of Eq. (4) depends essentially on the form of the comparison function $\Omega(\sigma)$. A reasonable approach consists in the elimination of σ from Eq. (4) and its reduction to a form, where the only unknown function is $(d\sigma/dx)$. The equation obtained for the case of the simple comparison function $\Omega^2(\sigma) = 1$ coincides with the Ermakov-Pinney equation, this will be explicitly shown in the next section. For more complicated forms of the comparison function $\Omega^2(\sigma)$ useful for the description of various physical problems starting with the classical problem of tunneling for non-relativistic quantum mechanics [1,2,10] and ending with the WKB-type analysis of the linearised equations for cosmological perturbations [11,12], the corresponding equation will acquire a more intricate form and we shall call it a generalised Ermakov-Pinney equation. The role of an unknown function in this equation is played by the function

$$y(x) \equiv \left(\frac{d\sigma}{dx}\right)^{-1/2} , \tag{5}$$

while the equation itself has the form

$$K\left(y,\frac{dy}{dx}\right)=\varepsilon L\left(y,\frac{dy}{dx},\frac{d^2y}{dx^2},\frac{d^3y}{dx^3}\right)\ , \tag{6}$$

where the concrete forms of the functions K and L will be defined in the next sections. We shall represent the function $y(x)$ as a series

$$y=\sum_{n=0}^{\infty}y_n\varepsilon^n\ , \tag{7}$$

and the general recurrence relation connecting the different coefficient functions y_n can be represented as

$$\left.\left(\frac{d^nK}{d\varepsilon^n}-\frac{d^{n-1}L}{d\varepsilon^{n-1}}\right)\right|_{\varepsilon=0}=0\ . \tag{8}$$

To implement this formula one can use the combinatorial relation [13]

$$\frac{d^n(uv)}{dx^n}=(u+v)^{(n)}\ , \tag{9}$$

where on the right-hand side one has a binomial expression wherein the powers are replaced by the derivatives. When the function $y(x)$ is constructed up to some level of approximation one can, in principle, find $\sigma(x)$ and substitute for it into Eq. (3).

We shall consider the implementation of this scheme for four choices of the comparison function: $\Omega^2(\sigma)=1,\Omega^2(\sigma)=\sigma,\Omega^2(\sigma)=\exp(\sigma)-1$ and $\Omega^2(\sigma)=\sigma^2-a^2$. At the end of the paper we shall discuss the relation between the different versions of the WKB method and equations of the Ermakov-Pinney type.

2 Differential equation without turning points

In the case for which the function $\omega^2(x)$ does not have zeros (turning points) it is convenient to choose the comparison function $\Omega^2(\sigma)=1$. In this case the function $U(\sigma)$ is simply an exponent, while Eq. (4) can be rewritten as

$$\varepsilon\frac{d^2y(x)}{dx^2}=\omega^2(x)y(x)-\frac{1}{y^3(x)}\ , \tag{10}$$

where the function $y(x)$ is defined by Eq. (5). The above equation is nothing but the well known Pinney or Ermakov-Pinney Eq. [8,9], which can be solved perturbatively with respect to the parameter ε [14]. We shall give here some general formulae for such a solution. It is convenient to rewrite Eq. (10) as

$$\omega^2 y^4 - 1 = \varepsilon y^3 \ddot{y} \ , \tag{11}$$

where a "dot" denotes the derivative with respect to x. We shall search for the solution of Eq. (11) in the form (7). The zero-order solution of Eq. (11) is

$$y_0 = \omega^{-1/2} \ , \tag{12}$$

and the general recurrence relation following from Eq. (11) for $n \geq 1$ is

$$\begin{aligned} y_n = & -\frac{1}{4y_0^3} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \sum_{k_3=0}^{n-1} \sum_{k_4=0}^{n-1} \delta\left[n - (k_1 + k_2 + k_3 + k_4)\right] y_{k_1} y_{k_2} y_{k_3} y_{k_4} \\ & + \frac{1}{4\omega^2 y_0^3} \sum_{k_1=0}^{n-1} \cdots \sum_{k_4=0}^{n-1} \delta\left[n - 1 - (k_1 + k_2 + k_3 + k_4)\right] \ddot{y}_{k_1} y_{k_2} y_{k_3} y_{k_4} \ , \end{aligned} \tag{13}$$

where $\delta()$ is the Kronecker delta symbol. It is now easy to obtain from the general expression (13) expressions for particular values of y_n

$$y_1 = \frac{\ddot{y}_0}{4\omega^2} \tag{14a}$$

$$y_2 = -\frac{6y_1^2}{4y_0} + \frac{\ddot{y}_1}{4\omega^2} + \frac{3\ddot{y}_0 y_1}{4\omega^2 y_0} \tag{14b}$$

$$\begin{aligned} y_3 = & -\frac{y_1^3}{y_0^2} - \frac{3y_2 y_1}{y_0} + \frac{\ddot{y}_2}{4\omega^2} \\ & + \frac{3\ddot{y}_0 y_2}{4\omega^2 y_0} + \frac{3\ddot{y}_1 y_1}{4\omega^2 y_0} + \frac{3\ddot{y}_0 y_1^2}{4\omega^2 y_0^2} \ . \end{aligned} \tag{14c}$$

Let us end by noting that the standard WKB approximation corresponds to the case discussed in this section and is of course only valid away from turning points.

3 Turning point: the Langer solution

In the case of the presence of a linear zero in the function $\omega^2(x)$ one can use the Langer solution [10]. In terms of the method of comparison equations

it means that one chooses the comparison function $\Omega^2(\sigma) = \sigma$. In this case Eq. (4) becomes

$$\omega^2(x) = \left(\frac{d\sigma}{dx}\right)^2 \sigma + \varepsilon \left(\frac{d\sigma}{dx}\right)^{1/2} \frac{d^2}{dx^2} \left(\frac{d\sigma}{dx}\right)^{-1/2} . \tag{15}$$

Dividing this equation by $(d\sigma/dx)^2$ and differentiating the result with respect to x we get an equation which depends only on the derivative $\dot{\sigma}$ and not on the function σ. Such an equation can be rewritten in the form

$$2\dot{\omega}\omega y^6 + 4\omega^2 \dot{y} y^5 = 1 + \varepsilon \left(3\ddot{y}\dot{y}y^4 + y^{(3)}y^5\right) , \tag{16}$$

where the function $y(x)$, as in the preceding section, is defined by Eq. (5). Again we shall search for the solution of Eq. (16) in the form (7). The equation for y_0 is

$$2\dot{\omega}\omega y_0^6 + 4\omega^2 \dot{y}_0 y_0^5 = \left(\omega^2 y_0^4\right)^{\cdot} y_0^2 = 1 . \tag{17}$$

This equation my be rewritten as

$$\sqrt{\omega^2 y_0^4} \, \frac{d}{dx} \left(\omega^2 y_0^4\right) = \omega , \tag{18}$$

and integrated

$$\omega^2 y_0^4 = \left(\frac{3}{2} \int \omega dx\right)^{2/3} . \tag{19}$$

Finally

$$y_0 = \frac{1}{\omega^{1/2}} \left(\frac{3}{2} \int \omega dx\right)^{\frac{1}{6}} . \tag{20}$$

Comparing the terms arising in Eq. (16) as coefficients of ε^n with $n \geq 1$ we have

$$4\omega^2 y_0^5 \dot{y}_n + \left(12\dot{\omega}\omega y_0^5 + 20\omega^2 \dot{y}_0 y_0^4\right) y_n = 4\omega^{-1/2} \left(\omega^3 y_0^5 y_n\right)^{\cdot} = F_n , \tag{21}$$

where F_n contains the terms which depend on $y_k, 0 \leq k \leq n-1$, and can be written as

$$
\begin{aligned}
F_n = & \sum_{k_1=0}^{n-1} \cdots \sum_{k_6=0}^{n-1} \Big\{ \delta\left[n-(k_1+k_2+k_3+k_4+k_5+k_6)\right] \left(4\omega^2 \dot{y}_{k_1} y_{k_2} - 2\dot{\omega}\omega y_{k_1} y_{k_2}\right) \\
& + \delta[n-1-(k_1+k_2+k_3+k_4+k_5+k_6)] \left(3\ddot{y}_{k_1}\dot{y}_{k_2} + y_{k_1}^{(3)} y_{k_2}\right) y_{k_3} y_{k_4} y_{k_5} y_{k_6} \Big\} \ . \qquad (22)
\end{aligned}
$$

The general expression for y_n is

$$
y_n = \frac{1}{4\omega^3 y_0^5} \int dx \omega F_n \ . \qquad (23)
$$

Let us also give the explicit expressions for the first terms of the expansion (7)

$$
y_1 = \frac{1}{4\omega^3 y_0^5} \int dx\omega \left(3\ddot{y}_0 \dot{y}_0 y_0^4 + y_0^{(3)} y_0^5\right) \qquad (24a)
$$

$$
\begin{aligned}
y_2 = & \frac{1}{4\omega^3 y_0^5} \int dx\omega \left(-30\dot{\omega}\omega y_1^2 y_0^4 - 20\omega^2 \dot{y}_1 y_1 y_0^4 - 40\omega^2 \dot{y}_0 y_1^2 y_0^4 \right. \\
& \left. + 3\ddot{y}_1 \dot{y}_0 y_0^4 + 3\ddot{y}_0 \dot{y}_1 y_0^4 + y_1^{(3)} y_0^5 + 5 y_0^{(3)} y_1 y_0^4\right) \qquad (24b)
\end{aligned}
$$

$$
\begin{aligned}
y_3 = & \frac{1}{4\omega^3 y_0^5} \int dx\omega \left(-60\dot{\omega}\omega y_2 y_1 y_0^4 - 40\dot{\omega}\omega y_1^3 y_0^3 - 20\omega^2 \dot{y}_2 y_1 y_0^4 \right. \\
& - 80\omega^2 \dot{y}_0 y_2 y_1 y_0^3 + 3\ddot{y}_2 \dot{y}_0 y_0^4 + 3\ddot{y}_0 \dot{y}_2 y_0^4 + 12\ddot{y}_0 \dot{y}_0 y_2 y_0^3 + 3\ddot{y}_1 \dot{y}_1 y_0^4 \\
& + 12\ddot{y}_1 \dot{y}_0 y_1 y_0^3 + 12\ddot{y}_0 \dot{y}_1 y_1 y_0^3 + 18\ddot{y}_0 \dot{y}_0 y_1^2 y_0^2 + y_2^{(3)} y_0^5 \\
& \left. + 5 y_0^{(3)} y_2 y_0^4 + 5 y_1^{(3)} y_1 y_0^4 + 10 y_0^{(3)} y_1^2 y_0^3\right) \ . \qquad (24c)
\end{aligned}
$$

Before closing this section we may compare our results with some formulae, obtained by Dingle [7]. In the vicinity of the turning point, which we choose as $x = 0$, the function $\omega^2(x)$ can be represented as

$$
\omega^2(x) = \gamma_1 x + \frac{\gamma_2}{2} x^2 + \frac{\gamma_3}{3!} x^3 + \frac{\gamma_4}{4!} x^4 + \cdots \qquad (25)
$$

while the function $\sigma(x)$ looks like

$$
\sigma(x) = \sigma_0 + \sigma_1 x + \frac{\sigma_2}{2} x^2 + \frac{\sigma_3}{3!} x^3 + \frac{\sigma_4}{4!} x^4 + \cdots \qquad (26)
$$

In reference [7] the first five coefficients $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ and σ_4, as functions of the coefficients $\gamma_1, \gamma_2, \gamma_3$ and γ_4, were obtained in the lowest approximation by using a system of recurrence relations. We can reproduce all these formulae by only using the general formula (20). Indeed, on writing

$$
\frac{d\sigma}{dx} = \frac{1}{y^2} = \frac{\omega(x)}{\left[\frac{3}{2}\int \omega(x)\right]^{1/3}} \qquad (27)
$$

using expansions (25) and (26) and comparing the coefficients of different powers of x we get the formulae

$$\begin{aligned}
\sigma_0 &= 0 \\
\sigma_1 &= \gamma_1^{1/3} \\
\sigma_2 &= \frac{1}{5}\gamma_1^{-2/3}\gamma_2 \\
\sigma_3 &= \frac{1}{7}\gamma_1^{-2/3}\gamma_3 - \frac{12}{175}\gamma_1^{-5/3}\gamma_2^2 \\
\sigma_4 &= \frac{1}{9}\gamma_1^{-2/3}\gamma_4 - \frac{44}{315}\gamma_1^{-5/3}\gamma_2\gamma_3 + \frac{148}{2625}\gamma_1^{-8/3}\gamma_2^3 \ ,
\end{aligned} \tag{28}$$

which coincide with those obtained in [7]. The value of σ_0 comes directly from Eq. (15) ignoring the ϵ-term and using the relations (25) and (26).

We can also find the ε-dependent corrections to the function $\sigma(x)$ in the neighbourhood of the point $x = 0$ by using the results (28) and the recurrence relations (24a) etc. We shall only give here the first correction, proportional to ε, to the coefficients σ_0 and σ_1. The value of σ_0 comes directly from Eq. (15) using the relations (25), (26) and (28)

$$\sigma_0 = \varepsilon \left(\frac{\gamma_3\gamma_1^{-5/3}}{14} - \frac{9\gamma_2^2\gamma_1^{-8/3}}{140} \right) \ . \tag{29}$$

In order to obtain σ_1 we introduce into the definition of the function $y(x)$, Eq. (5), the expansion (26) and

$$y_0 = \sigma_1^{-1/2} \left[1 - \frac{\sigma_2}{2\sigma_1}x + \left(\frac{3\sigma_2^2}{8\sigma_1^2} - \frac{\sigma_3}{4\sigma_1} \right) x^2 + \left(\frac{3\sigma_2\sigma_3}{8\sigma_1^2} - \frac{\sigma_4}{12\sigma_1} - \frac{5\sigma_2^3}{16\sigma_1^3} \right) x^3 \right] + \cdots \tag{30}$$

Now, on using the recurrence formula (24a) we find the leading (constant) term in the expression for y_1

$$y_1 = \frac{1}{\gamma_1} \left(-\frac{1}{2}\sigma_2^3\sigma_1^{-7/2} + \frac{1}{2}\sigma_3\sigma_2\sigma_1^{-5/2} - \frac{\sigma_4\sigma_1^{-3/2}}{12} \right) + \cdots \tag{31}$$

From the formula

$$\frac{d\sigma}{dx} = \frac{1}{y^2} = \frac{1}{(y_0 + \varepsilon y_1)^2} \tag{32}$$

it is easy to find the first correction to the first coefficient σ_1 in the expansion (26)

$$\sigma_1 = \gamma_1^{1/3} - 2\varepsilon \frac{y_1}{y_0^3} \ . \tag{33}$$

Substituting into Eq. (33) the leading term of y_1 from Eq. (31), of y_0 from (30) and using the explicit expressions for the coefficients σ_i from Eq. (28) we finally obtain

$$\sigma_1 = \gamma_1^{1/3} + \varepsilon \left(\frac{7}{225} \gamma_2^3 \gamma_1^{-11/3} - \frac{7}{135} \gamma_3 \gamma_2 \gamma_1^{-8/3} + \frac{1}{54} \gamma_4 \gamma_1^{-5/3} \right) \ . \tag{34}$$

The results (29) and (34) was found by Dingle [7], but in his article there were two sign errors in the second line of equation (69).

4 Turning point: more complicated comparison function

Let us consider the differential Eq. (1), when the function $\omega^2(x)$ has a linear turning point, but instead of Langer's comparison function $\Omega^2(\sigma) = \sigma$ consider a more complicated comparison function, also having linear turning point, namely

$$\Omega^2(\sigma) = e^\sigma - 1 \ . \tag{35}$$

Now, Eq. (4) will have the form

$$\omega^2(x) = \left(\frac{d\sigma}{dx} \right)^2 (e^\sigma - 1) + \varepsilon \left(\frac{d\sigma}{dx} \right)^{1/2} \frac{d^2}{dx^2} \left(\frac{d\sigma}{dx} \right)^{-1/2} \ . \tag{36}$$

Isolating the function e^σ, taking its logarithm and differentiating it with respect to x we arrive to the equation which involves only the derivative $\dot{\sigma}$. This equation can be written down as

$$\omega^2 y^4 - 2\dot{\omega}\omega y^6 - 4\omega^2 \dot{y} y^5 + 1 = \varepsilon \left(\ddot{y} y^3 - 3\ddot{y}\dot{y} y^4 - y^{(3)} y^5 \right) \ . \tag{37}$$

To find the function y_0 one should solve the equation

$$\omega^2 y_0^4 - 2\dot{\omega}\omega y_0^6 - 4\omega^2 \dot{y}_0 y_0^5 + 1 = 0 \ . \tag{38}$$

On introducing the new variable

$$z \equiv \omega^2 y_0^4 \tag{39}$$

one can rewrite Eq. (38) as

$$\dot{z} = \frac{(z+1)\omega}{z^{1/2}} \ . \tag{40}$$

It is also convenient to introduce the variable

$$v \equiv \sqrt{z} \ . \tag{41}$$

Eq. (40) now becomes

$$\frac{2v^2\dot{v}}{1+v^2} = \omega \tag{42}$$

which can be integrated, obtaining

$$2v - 2\arctan v = \int \omega dx \ . \tag{43}$$

Finally, for y_0 we have the implicit representation

$$y_0^2 - \frac{1}{\omega}\arctan \omega y_0^2 = \frac{1}{2\omega}\int \omega dx \ . \tag{44}$$

We may now write the equation defining the recurrence relation for $y_n, n \geq 1$

$$-4\omega^2 y_0^5 \dot{y}_n + \left(4\omega^2 y_0^3 - 20\omega^2 \dot{y}_0 y_0^4 - 12\dot{\omega}\omega y_0^5\right) y_n = F_n, \tag{45}$$

where

$$
\begin{aligned}
F_n = & -\omega^2 \sum_{k_1=0}^{n-1} \cdots \sum_{k_4=0}^{n-1} \delta\left[n-(k_1+k_2+k_3+k_4)\right] y_{k_1} y_{k_2} y_{k_3} y_{k_4} \\
& +2\dot{\omega}\omega \sum_{k_1=0}^{n-1} \cdots \sum_{k_6=0}^{n-1} \delta\left[n-(k_1+k_2+k_3+k_4+k_5+k_6)\right] y_{k_1} \cdots y_{k_6} \\
& +4\omega^2 \sum_{k_1=0}^{n-1} \cdots \sum_{k_6=0}^{n-1} \delta\left[n-(k_1+k_2+k_3+k_4+k_5+k_6)\right] \dot{y}_{k_1} y_{k_2} \cdots y_{k_6} \\
& + \sum_{k_1=0}^{n-1} \cdots \sum_{k_4=0}^{n-1} \delta\left[n-1-(k_1+k_2+k_3+k_4)\right] \ddot{y}_{k_1} y_{k_2} y_{k_3} y_{k_4} \\
& -3 \sum_{k_1=0}^{n-1} \cdots \sum_{k_6=0}^{n-1} \delta\left[n-1-(k_1+k_2+k_3+k_4+k_5+k_6)\right] \ddot{y}_{k_1} \dot{y}_{k_2} y_{k_3} \cdots y_{k_6} \\
& - \sum_{k_1=0}^{n-1} \cdots \sum_{k_6=0}^{n-1} \delta\left[n-1-(k_1+k_2+k_3+k_4+k_5+k_6)\right] y_{k_1}^{(3)} y_{k_2} \cdots y_{k_6} \ . \qquad (46)
\end{aligned}
$$

The solution for y_n is

$$
y_n = -\frac{1}{4\omega^3 y_0^5 G} \int dx\, \omega G F_n \ , \qquad (47)
$$

where

$$
G(x) = \exp\left(-\int^x \frac{dx'}{y_0^2(x')} \right) \ . \qquad (48)
$$

Here we give the explicit expressions for the first two functions F_1 and F_2

$$
\begin{aligned}
F_1 = & \ \ddot{y}_0 y_0^3 - 3\ddot{y}_0 \dot{y}_0 y_0^4 - y_0^{(3)} y_0^5 \ , \qquad (49a) \\
F_2 = & -5\omega^2 y_1^2 y_0^2 + 30\dot{\omega}\omega y_1^2 y_0^4 + 20\omega^2 \dot{y}_1 y_1 y_0^4 + 40\omega^2 \dot{y}_0 y_1^2 y_0^3 + \ddot{y}_1 y_0^5 \\
& + 5\dot{y}_0 y_1 y_0^4 - 3\ddot{y}_1 \dot{y}_0 y_0^4 - 3\ddot{y}_0 \dot{y}_1 y_0^4 - 12\ddot{y}_0 \dot{y}_0 y_1 y_0^3 - y_1^{(3)} y_0^5 - 5y_0^{(3)} y_1 y_0^4 \ . \qquad (49b)
\end{aligned}
$$

5 Two turning points

Let us now consider the differential Eq. (1), when the function $\omega^2(x)$ has two turning points. This situation, describing the tunneling through a potential barrier was considered in [6,2]. In this case the comparison function can be chosen as

$$
\Omega^2(\sigma) = \sigma^2 - a^2 \ . \qquad (50)
$$

The solutions $U(\sigma)$ of Eq. (2) are the well-known parabolic functions [2]. Substituting $\Omega^2(\sigma)$ of Eq. (50) into Eq. (4) one can isolate σ and after the subsequent differentiation one can write down the equation for the function $y(x)$

$$2\sqrt{\omega^2 y^4 + a^2 + \varepsilon \ddot{y} y^3} = 2\dot{\omega}\omega y^6 + 4\omega^2 \dot{y} y^5 + \varepsilon \left(3\ddot{y}\dot{y}y^4 + y^{(3)} y^5 \right) \ . \tag{51}$$

The lowest-approximation y_0 can be found from the equation

$$2\sqrt{\omega^2 y^4 + a^2} = 2\dot{\omega}\omega y^6 + 4\omega^2 \dot{y} y^5 \ . \tag{52}$$

This equation can be integrated and the solution y_0 can be written down in the implicit form

$$\omega y_0^3 - \frac{a^2}{\omega} \mathrm{arcsinh} \frac{\omega y_0}{a} = \frac{2}{\omega} \int \omega dx \ . \tag{53}$$

To get the recurrence relations for the functions $y_n, n \geq 1$ it is convenient to take the square of Eq. (51)

$$\begin{aligned} &4\dot{\omega}^2\omega^2 y^{12} + 16\omega^4 \dot{y}^2 y^{10} + 16\dot{\omega}\omega^3 \dot{y} y^{11} - 4\omega^2 y^4 - 4a^2 = \\ &\quad \varepsilon(4\ddot{y}y^3 - 12\dot{\omega}\omega\ddot{y}\dot{y}y^{10} - 24\omega^2\ddot{y}\dot{y}^2 y^9 - 4\dot{\omega}\omega y^{(3)} y^{11} - 8\omega^2 y^{(3)} \dot{y} y^{10}) \\ &\quad -\varepsilon^2 (9\ddot{y}^2 \dot{y}^2 y^8 + {y^{(3)}}^2 y^{10} + 6 y^{(3)} \ddot{y}\dot{y} y^9) \ . \end{aligned} \tag{54}$$

On comparing the terms containing the n^{th} powers of the small parameter ε one obtains

$$f\dot{y}_n + g y_n = F_n \ , \tag{55}$$

where

$$
\begin{aligned}
f &= 32\omega^4 \dot{y}_0 y_0^{10} + 16\dot{\omega}\omega^3 y_0^{11} &\text{(56a)}\\
g &= 48\dot{\omega}^2\omega^2 y_0^{11} + 160\omega^4 \dot{y}_0^2 y_0^9 + 176\dot{\omega}\omega^3 \dot{y}_0 y_0^{10} - 16\omega^2 y_0^3 &\text{(56b)}\\
F_n &= \sum_{k_1=0}^{n-1}\cdots\sum_{k_{12}=0}^{n-1}\delta\left[n-(k_1+\cdots+k_{12})\right]\\
&\quad\times 4\left(\dot{\omega}^2\omega^2 y_{k_1}y_{k_2} + 16\omega^4\dot{y}_{k_1}\dot{y}_{k_2} + 16\dot{\omega}\omega^3\dot{y}_{k_1}y_{k_2}\right)y_{k_3}\cdots y_{k_{12}}\\
&\quad-4\omega^2\sum_{k_1=0}^{n-1}\cdots\sum_{k_4=0}^{n-1}\delta\left[n-(k_1+\cdots+k_4)\right]y_{k_1}\cdots y_{k_4}\\
&\quad-\varepsilon\left\{-4\sum_{k_1=0}^{n-1}\cdots\sum_{k_4=0}^{n-1}\delta\left[n-1-(k_1+\cdots+k_4)\right]\ddot{y}_{k_1}y_{k_2}y_{k_3}y_{k_4}\right.\\
&\quad+\sum_{k_1=0}^{n-1}\cdots\sum_{k_{12}=0}^{n-1}\delta\left[n-1-(k_1+\cdots+k_{12})\right]\\
&\qquad\qquad\times\left(12\dot{\omega}\omega\ddot{y}_{k_1}\dot{y}_{k_2}y_{k_3} + 24\omega^2\ddot{y}_{k_1}\dot{y}_{k_2}\dot{y}_{k_3}\right.\\
&\qquad\qquad\left.\left.+4\dot{\omega}\omega y^{(3)}y_{k_2}y_{k_3} + 8\omega^2 y_{k_1}^{(3)}\dot{y}_{k_2}y_{k_3}\right)y_{k_4}\cdots y_{k_{12}}\right\}\\
&\quad-\varepsilon^2\sum_{k_1=0}^{n-2}\cdots\sum_{k_{12}=0}^{n-2}\delta\left[n-2-(k_1+\cdots+k_{12})\right]\\
&\quad\times\left(9\ddot{y}_{k_1}\ddot{y}_{k_2}\dot{y}_{k_3}\dot{y}_{k_4} + y_{k_1}^{(3)}y_{k_2}^{(3)}y_{k_3}y_{k_4} + 6y_{k_1}^{(3)}\dot{y}_{k_2}y_{k_3}y_{k_4}\right)y_{k_5}\cdots y_{k_{12}}\ . &\text{(56c)}
\end{aligned}
$$

The integration of Eq. (55) gives

$$
y_n = \frac{1}{G}\int dx\frac{GF_n}{f}\ , \tag{57}
$$

where

$$
G = \exp\left(\int dx'\frac{g}{f}\right)\ . \tag{58}
$$

The explicit expressions for the coefficients F_n are rather cumbersome and we shall only write down the coefficient F_1

$$
\begin{aligned}
F_1 &= 48\dot{\omega}^2\omega^2 y_1 y_0^{11} + 32\dot{y}_1\dot{y}_0 y_0^{10} + 160\omega^4\dot{y}_0^2 y_1 y_0^9\\
&\quad+16\dot{\omega}\omega^3\dot{y}_1 y_0^{11} + 176\dot{\omega}\omega^3\dot{y}_0 y_1 y_0^{10} - 16\omega^2 y_1 y_0^3 + 4\ddot{y}_0 y_0^3\\
&\quad-12\dot{\omega}\omega\ddot{y}_0\dot{y}_0 y_0^{10} - 24\ddot{y}_0\dot{y}_0^2 y_0^9 - 4\dot{\omega}\omega y_0^{(3)}y_0^{11} - 8\omega^2 y_0^{(3)}\dot{y}_0 y_0^{10}\ .
\end{aligned}
\tag{59}
$$

6 Discussion: WKB method and the generalized Ermakov-Pinney equation

In this paper we have studied the relation between the WKB method for the solution of second-order differential equations and the Ermakov-Pinney equation. For some non-trivial applications of the WKB method it is found that instead of the standard form of the Ermakov- Pinney equation one is lead to its generalizations which are rather different to the known generalizations of Ermakov systems.

Let us note that although the Ermakov equation was already reproduced in the WKB context (first of all in the important paper by Milne [15]), its generalizations arising in the process of application of the comparison function method [6,7] were not yet studied, at least to the best of our knowledge. In this concluding section of our paper we shall try to give a short review of the history of the Ermakov-Pinney equation and its applications to quantum mechanics and to similar WKB-type problems and discuss the physical significance of its non-trivial generalization (for a more detailed account of the history of the Eramkov-Pinney equation see e.g. [16]).

The relation between the second-order linear differential equation

$$\frac{d^2u}{dx^2} + \omega^2(x)u(x) = 0 \tag{60}$$

and the non-linear differential equation

$$\frac{d^2\rho}{dx^2} + \omega^2(x)\rho(x) = \frac{\alpha}{\rho^3(x)} \ , \tag{61}$$

where α is some constant was noticed by Ermakov [8], who showed that any two solutions ρ and u of the above equations are connected by the formula

$$C_1 \int \frac{dx}{u^2} + C_2 = \sqrt{C_1 \frac{\rho^2}{u^2} - \alpha} \ . \tag{62}$$

The couple of Eqs. (60) and (61) constitute the so called Ermakov system. An important corollary was derive [8] from the formula (62). Namely, on having a particular solution $\rho(x)$ of Eq. (61) one can construct the general solution of Eq. (60) which is given by

$$u(x) = c_1\rho(x)\exp\left[\sqrt{-\alpha}\int \frac{dx}{\rho^2(x)}\right] + c_2\rho(x)\exp\left[-\sqrt{-\alpha}\int \frac{dx}{\rho^2(x)}\right] \ . \tag{63}$$

It is easy to see that the function $\rho(x)$ plays of the role of an "amplitude" of the function $u(x)$, while the integral $\int [dx/\rho^2(x)]$ represents some kind of "phase". Thus, it is not surprising that the Ermakov equation was re-discovered by Milne [15] in a quantum-mechanical context. On introducing the amplitude, obeying Eq. (61) and the phase, Milne constructed a formalism for the solution of the Schrödinger equation which was equivalent to the WKB method. Milne's version of the WKB technique was extensively used for the solution of quantum mechanical problems (see e.g. [17]). In the paper by Pinney [9] the general form of the solution of the Ermakov equation was presented, while the most general expression for this solution was written down by Lewis [14]. The equation, sometimes called Ermakov-Milne-Pinney equation, has also found application in cosmology [18].

A very simple physical example rendering trasparent the derivation of the Ermakov-Pinney equation and its solutions was given in the paper by Eliezer and Gray [19]. The motion of a two-dimensional oscillator with time-dependent frequency was considered. In this case the second-order linear differential Eq. (60) describes the projection of the motion of this two-dimensional oscillator on one of its Cartesian coordinates, while the Ermakov-Pinney Eq. (61) describes the evolution of the radial coordinate ρ. The parameter α is nothing more than the squared conserved angular momentum of the two-dimensional oscillator. Thus, the notion of the amplitude and phase acquire in this example a simple geoemtrical and physical meaning.

An additional generalization of the notion of Ermakov system of equations was suggested in the paper by Ray and Reid [20]. Thy consider the system of two equations

$$\frac{d^2u}{dx^2} + \omega^2(x)u(x) = \frac{1}{\rho x^2} g\left(\frac{\rho}{x}\right) \tag{64}$$

and

$$\frac{d^2\rho}{dx^2} + \omega^2(x)\rho(x) = \frac{1}{x\rho^2(x)} f\left(\frac{x}{\rho}\right) \ , \tag{65}$$

where g and f are arbitrary functions of their arguments. The standard Ermakov system (60), (61) corresponds to the choice of functions $g(\xi) = 0$ and $f(\xi) = \alpha\xi$. One can show that the generalized Ermakov system (64), (65) has an invariant (zero total time derivative)

$$I_{f,g} = \frac{1}{2}\left[\phi\left(\frac{u}{\rho}\right) + \theta\left(\frac{\rho}{u}\right) + \left(u\frac{d\rho}{dx} - \rho\frac{du}{dx}\right)^2\right] \ , \tag{66}$$

where

$$\phi\left(\frac{u}{\rho}\right) = 2\int^{\frac{u}{\rho}} f(x)dx \tag{67a}$$

$$\theta\left(\frac{\rho}{u}\right) = 2\int^{\frac{\rho}{u}} g(x)dx \ . \tag{67b}$$

The invariant (66) establishes the connection between the solutions of Eqs. (64) and (65) and sometimes allows one to find the solution of one of these equations provided the solution of the other one is known (just as in the case of the standard Ermakov system). However, for the case of an arbitrary couple of functions f and g a simple physical or geometrical interpretation of the Ermakov system analogous to that given in [19] is not known.

In the present paper we, on one hand, have established the connection between the generalized WKB method (method of comparison Eqs. [6,7]) and the Ermakov - Pinney equation and on the other hand we have obtained a generalization of the Ermakov - Pinney equation, different to those studied in the literature. Let us briefly summarize our approach. One has two second-order differential Eqs. (1) and (2) and the solution of one of them (2) is well known and described. Using the previous convenient analogy [19], one can say that in this case one has two time-dependent oscillators, evolving with two different time-parameters. One can try to find the solution of Eq. (1) by representing it as a known solution of Eq. (2) multiplied by a correction factor. This correction factor plays the role of the prefactor in the standard WKB approach while the known solution of Eq. (2) represents some kind of generalized phase term. Further, the prefactor is expressed in terms of the derivative between two variables x and σ (or in terms of the oscillator analogy, between two times). On writing down the equation definining this factor, which we have denoted by $y(x)$ (see Eq. (5)) we arrive to the Eq. (4) which could be called generalized Ermakov-Pinney equation. For the case when the function $\omega^2(x)$ does not have turning points the comparison function $\Omega^2(\sigma)$ can be chosen constant (the second oscillator has a time-independent frequency) and the the equation defining the prefactor becomes the standard Ermakov-Pinney Eq. (10). In terms of the two-dimensional oscillator analogy [19] this means that we exclude from the equation for the radial coordinate the dependence on the angle coordinate σ by using its cyclicity, i.e. the conservation of the angular moment.

In cases for which the function $\omega^2(x)$ has turning points, as in Secs. 3, 4 and 5, the comparison functions are non-constant and instead of the standard Ermakov-Pinney equation we have its non-trivial generalizations. The common feature of these generalizations consists in the fact that the corresponding equations for the variable y depend explicitly also on the parameter σ which is not excluded automatically. To get a differential equation for y, one should

isolate the parameter σ and subsequently differentiate with respect to x. As a result one gets a differential equation of higher-order for the function $y(x)$. Remarkably, for the perturbative solution of these equations one can again construct a reasonable iterative procedure.

Again, it is interesting to look at the generalized Ermakov-Pinney equation as an equation describing a two-dimensional physical system in the spirit of the reference [19]. In our case one has

$$\frac{d^2y(x)}{dx^2} + \omega^2(x)y(x) = \frac{\Omega^2(\sigma)}{y^3(x)} , \tag{68}$$

where y plays the role of a radial coordinate, σ resembles a phase or an angle (e.g. the position of a hand of a clock) and x is a time parameter. It is important to notice that according to the definition of the variable y (5) there is a relation between the radial and angle coordinates

$$\frac{d\sigma}{dx} = \frac{1}{y^2} , \tag{69}$$

which on interpreting σ as a phase corresponds to constant (unit) angular momentum. In our case however the right-hand side of the radial Eq. (68) contains an explicit dependence on the "angle" σ. Thus, the couple of Eqs. (68) and (69) cannot be represented as a system of equations of motion corresponding to some Lagrangian or Hamiltonian as in Ref. [19]. Perhaps, this system could be described in terms of non-Hamiltonian dissipative dynamics, but this question requires a further study.

References

[1] L.I. Schiff, Quantum mechanics, McGraw Hill, New York, 1968.

[2] M.V. Berry and K.E. Mount, Rept. Prog. Phys. 35 (1972) 315.

[3] B.S. DeWitt, Dynamical theory of groups and fields, Gordon and Breach, New York, 1965.

[4] F.W.J. Olver, Asymptotics and special functions, Academic Press, New York, 1974.

[5] R.B. Dingle, Asymptotic expansions: their derivation and interpretation, Academic Press, London, 1973.

[6] S.C. Miller Jr. and R.H. Good Jr., Phys. Rev. 91 (1953) 174.

[7] R. B. Dingle, Appl. Sci. Res. B 5 (1956) 345.

[8] V.P. Ermakov, Univ. Izv. Kiev, series III 9 (1880) 1.

[9] E. Pinney, Proc. Amer. Math. Soc. 1 (1950) 681.

[10] R.E. Langer, Phys. Rev. 51 (1937) 669.

[11] J. Martin and D.J. Schwarz, Phys. Rev. D 67 (2003) 083512; S. Habib, A. Heinen, K. Heitmann, G. Jungman and C. Molina-Paris, Phys. Rev. D 70 (2004) 083507; R. Casadio, F. Finelli, M. Luzzi and G. Venturi, Phys. Rev. D 71 (2005) 043517.

[12] R. Casadio, F. Finelli, A. Kamenshchik, M. Luzzi and G. Venturi, arXiv: ?????????; R. Casadio, F. Finelli, M. Luzzi and G. Venturi, Phys. Rev. D 72 (2005) 103516; R. Casadio, F. Finelli, M. Luzzi and G. Venturi, Phys. Lett. B (2005) 1

[13] I.S. Gradstein and I.M. Ryzhik, Tables of integrals, series and products, Academic Press, New York, 1980.

[14] H. R. Lewis, J. Math. Phys. 9 (1968) 1976.

[15] W.E. Milne, Phys. Rev. 35 (1930) 863.

[16] P. Espinosa, math-ph/0002005.

[17] U. Fano and A.R.P. Rau, Atomic Collisions and Spectra, Academic Press, Orlando, 1986.

[18] F. Finelli, G.P. Vacca and G. Venturi, Phys. Rev. D 58 (1998) 103514; C. Bertoni, F. Finelli and G. Venturi, Phys. Lett. A 237 (1998) 331; F. Finelli, A. Grupposo and G. Venturi, Class. Quantum Grav. 16 (1999) 3923; H. Rosu, P. Espinosa and M. Reyes, Nuovo Cim. B 114 (1999) 1435; R.M. Hawkins and J.E. Lidsey, Phys. Rev. D 66 (2002) 023523.

[19] C. J. Eliezer and A. Gray, SIAM J. Appl. Math. 30 (1976) 463.

[20] J. R. Ray and J.L. Reid, Phys. Lett. A 71 (1979) 317.